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Tangential Characteristics and Coupling of Waves*

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1. INTRODUCTION

For a linear hyperbolic system of partial differential equations, it can happen that two propagation speeds coalesce at a surface S in space-time. A corresponding pair of characteristic surfaces will be tangent at S, and their propagation modes may be coupled. In the present work, highly oscillatory solutions are considered near such a surface S, and the leading terms of the solution are described. The analysis includes an asymptotic estimate of a double integral, which is of independent interest. An application is given to the Lundquist equations of magnetohydrodynamics, in a case where the Alfvèn and sound speeds coalesce.

We consider the equation

$$\mathscr{L}v \equiv A^0(x)\, v_t + \sum_{\mu=1}^{n} A^\mu(x) \frac{\partial v}{\partial x^\mu} + B(t, x)\, v = 0, \tag{1.1}$$

where v is a vector with m components, A^0 and A^μ ($\mu = 1,\dots n$) are $m \times m$ matrices which depend smoothly on x, and A^0 is non-singular. The matrix B depends smoothly on t and x. The characteristic equation for (1.1) is

$$\det\left(A^0\lambda + \sum_{\mu=1}^{n} A^\mu p_\mu\right) = 0. \tag{1.2}$$

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It is assumed that the roots $\lambda^j(x, p)$ of (1.2) ($j = 1,..., m$) are all real for real $p \neq 0$, and that the associated null eigenvectors of the matrix in (1.2) form a complete set. It is further assumed that the roots are distinct, except that

$$\lambda^1(x, p) = \lambda^2(x, p) \qquad \text{for } x^1 = 0, \tag{1.3}$$

identically in the remaining variables, and that

$$\lambda^1_{p_1}(x, p)\,(\lambda^1_{x^1}(x, p) - \lambda^2_{x^1}(x, p)) \neq 0 \qquad \text{for } x^1 = 0. \tag{1.4}$$

The latter two conditions imply that the propagation velocities cross at the surface $S : x^1 = 0$, but the rays associated with λ^1 (and λ^2) are not tangent to S.

Initial data for (1.1) are prescribed in the form of an asymptotic series

$$v(0, x) = e^{ik\phi(x)} \sum_{l=0}^{\infty} \frac{c_l(x)}{(ik)^l}, \tag{1.5}$$

where ϕ and c_l are smooth functions. For simplicity, we also assume that $c_l(x)$ vanish near S. The main result of the present work is that for $x^1 \neq 0$ the solution has the form

$$\begin{aligned} v(t, x) = \sum_{j=1}^{m} &\exp(ik\phi^j(t, x))\, c_0{}^j(t, x) \\ &+ R_{12} \exp(ik\phi^{12}(t, x))\, d^{12}(t, x) \\ &+ R_{21} \exp(ik\phi^{21}(t, x))\, d^{21}(t, x) + 0\left(\frac{\log k}{k}\right). \end{aligned} \tag{1.6}$$

This result holds formally under the conditions stated above, and it is proven to hold in L_2 sense when A^l ($l = 0, 1,..., n$) are symmetric. The terms after the summation sign in (1.6) are given by geometrical optics: $\phi^j(t, x)$ are solutions of the characteristic equation with initial data given by $\phi(x)$, and $c_0{}^j$ are the associated transport coefficients (see below). The functions $\phi^{\alpha\beta}$ are obtained by drawing the ray corresponding to λ^β from (t, x) back to the surface S, and from there drawing the ray corresponding to λ^α back to the initial surface. The coefficients $R_{\alpha\beta}$ are diffraction coefficients which arise from splitting of rays at S, and $d^{\alpha\beta}(t, x)$ are transport coefficients. A more complicated form of the solution is valid near S, or if the initial data do not vanish near S.

In Section 2, the solution of (1.1) and (1.5) is constructed in the form of an asymptotic series, motivated by the work of Ludwig and Granoff [7]. The present case is more complicated, since the higher terms in the expansion involve multiple integrals of successively higher order. In Section 3, the critical points of the single and double integrals are discussed. This information is used to estimate the double integrals in Section 4. In Section 5, the leading terms in the expansion are simplified to obtain (1.6). Finally, the application to MHD is given in Section 6. The authors are indebted to H. Kranzer for suggesting this application.

2. Construction of the Asymptotic Solution

The usual geometrical optics solution is presented first, in order to define the notation and indicate the nature of the difficulty where $x^1 = 0$. Then Duhamel's principle or the method of Ludwig and Granoff [7] suggests a modification of the form of the solution, which involves multiple integrals. The coefficients in the more complicated expansion are determined by a recursive procedure, similar to that for geometrical optics.

We first attempt to find a solution of (1.1) in the same form as the initial data:

$$v(t, x) = \sum_{j=1}^{m} \left[\exp(ik\phi^j(t, x)) \sum_{l=0}^{\infty} \frac{c_l^j(t, x)}{(ik)^l} \right]. \tag{2.1}$$

After inserting (2.1) into (1.1) and grouping terms for each j according to powers of k, we obtain

$$A_j c_0^j = 0, \tag{2.2}$$

$$A_j c_{l+1}^j + \mathscr{L}(c_l^j) = 0, \qquad (l = 0, 1, \ldots). \tag{2.3}$$

Here we have introduced the notation

$$A_j = A(\phi^j) = A^0 \phi_t^j + \sum_{\mu=1}^{n} A^\mu \frac{\partial \phi^j}{\partial x^\mu}. \tag{2.4}$$

Equation (2.2) implies (1.2), or

$$\phi_t^j = \lambda^j(x, \nabla\phi^j), \tag{2.5}$$

and also implies that A_j has left and right null vectors $L^j(x, \nabla\phi^j)$ and $R^j(x, \nabla\phi^j)$:

$$L^j A_j = 0, \qquad A_j R^j = 0. \tag{2.6}$$

After multiplication by L^j, (2.3) becomes, for $l = 0$,

$$L^j \mathscr{L}(c_0^j) = 0. \tag{2.7}$$

This is a scalar equation for the magnitude of $c_0{}^j$, since $c_0{}^j$ must be a multiple of R^j. Initial values for ϕ^j and $c_0{}^j$ are obtained from (1.5). Then (2.5) and (2.7) may be solved by integration along the associated ray system:

$$\frac{dx^i}{ds} = -\frac{\partial\lambda^j(x, p)}{\partial p_i}, \qquad \frac{dt}{ds} = 1. \tag{2.8}$$

$$\frac{dp_i}{ds} = \frac{\partial\lambda^j(x, p)}{\partial x^i}. \tag{2.9}$$

Further details are given in Courant [2, Chap. VI]. The next coefficients $c_1{}^j$ are determined modulo R^j by solving (2.3). The compatibility condition for the system is (2.7).

The following difficulty arises at $x^1 = 0$, where $\lambda^1 = \lambda^2$: the rank of A_1 is $m - 1$ for $x^1 \neq 0$, but it drops to $m - 2$ at $x^1 = 0$. Therefore, there is an additional compatibility condition to be satisfied at $x^1 = 0$, namely

$$L^2\mathscr{L}(c_0{}^1) = 0. \tag{2.10}$$

In general, this is not consistent with (2.7) with $j = 1$, and $c_0{}^1$ becomes infinite at $x^1 = 0$. A similar difficulty appears in Ludwig and Granoff [7]. The solution given there suggests the following form of v:

$$\begin{aligned} v(t, x) = \sum_{l=0}^{\infty} \sum_{j=1}^{m} \exp(ik\phi^j) \frac{c_l{}^j}{(ik)^l} \\ + \sum_{\substack{\alpha=1 \\ \beta=3-\alpha}}^{2} \Bigg\{ \int_0^t \exp(ik\phi^{\alpha\beta}) \frac{c_l^{\alpha\beta}(t, x; \tau)}{(ik)^l}\, d\tau \\ + \int_0^t \int_0^\sigma \exp(ik\phi^{\alpha\beta\alpha}) \frac{c_l^{\alpha\beta\alpha}(t, x; \sigma, \tau)}{(ik)^l}\, d\tau\, d\sigma + \cdots \Bigg\}. \end{aligned} \tag{2.11}$$

Here $\phi^{\alpha\beta}(t, x; \tau)$ and $\phi^{\alpha\beta\alpha}(t, x; \sigma, \tau)$ satisfy

$$\phi_t^{\alpha\beta} = \lambda^\beta(x, \nabla\phi^{\alpha\beta}), \tag{2.12}$$

$$\phi^{\alpha\beta}(\tau, x; \tau) = \phi^\alpha(\tau, x), \tag{2.13}$$

$$\phi_t^{\alpha\beta\alpha}(t, x; \sigma, \tau) = \lambda^\alpha(x, \nabla\phi^{\alpha\beta\alpha}) \tag{2.14}$$

$$\phi^{\alpha\beta\alpha}(\sigma, x; \sigma, \tau) = \phi^{\alpha\beta}(\sigma, x; \tau). \tag{2.15}$$

After substituting (2.11)–(2.15) into (1.1) and collecting terms with the same power of k, and the same exponential factor, we obtain the conditions

$$A(\phi^\alpha)\, c_{l+1}^\alpha + \mathscr{L}(c_l^\alpha) + c_l^{\alpha\beta}(t, x; t) = 0, \tag{2.16}$$

$$A(\phi^{\alpha\beta})\, c_{l+1}^{\alpha\beta} + \mathscr{L}(c_l^{\alpha\beta}) + c_l^{\alpha\beta\alpha}(t, x; t, \tau) = 0, \tag{2.17}$$

for $l = -1, 0, 1, \ldots$, $\alpha = 1, 2$, and $\beta = 3 - \alpha$.

The final terms in (2.16) and (2.17) enable us to impose an additional compatibility contition $\mathscr{L}(c_0^\alpha)$, etc.: we require that

$$L^\alpha(x, \nabla\sigma^\alpha)\, [\mathscr{L}(c_0^\alpha(t, x)) + c_0^{\alpha\beta}(t, x; t)] = 0, \tag{2.18}$$

$$L^\beta(x, \nabla\phi^\alpha)\, [\mathscr{L}(c_0^\alpha(t, x)) + c_0^{\alpha\beta}(t, x; t)] = 0, \tag{2.19}$$

$$L^\beta(x, \nabla\phi^{\alpha\beta})\, [\mathscr{L}(c_0^{\alpha\beta}(t, x; \tau)) + c_0^{\alpha\beta\alpha}(t, x; t, \tau)] = 0, \tag{2.20}$$

$$L^\alpha(x, \nabla\phi^{\alpha\beta})\, [\mathscr{L}(c_0^{\alpha\beta}(t, x; \tau)) + c_0^{\alpha\beta\alpha}(t, x; t, \tau)] = 0. \tag{2.21}$$

If these equations are satisfied, then c_1^α, $c_1^{\alpha\beta}$, etc., will be regular where $x^1 = 0$. We note from (2.17) with $l = -1$ that $c_0^{\alpha\beta}$ must be proportional to $R^\beta(x, \nabla\phi^{\alpha\beta})$;

$$c_0^{\alpha\beta}(t, x; \tau) = \gamma_0^{\alpha\beta}(t, x; \tau)\, R^\beta(x, \nabla\phi^{\alpha\beta}). \tag{2.22}$$

It follows from differentiation of (2.13) that

$$\nabla\phi^{\alpha\beta}(t, x; t) = \nabla\phi^\alpha(t, x), \tag{2.23}$$

and therefore

$$R^\beta(x, \nabla\phi^{\alpha\beta}) = R^\beta(x, \nabla\phi^\alpha), L^\beta(x, \nabla\phi^{\alpha\beta}) = L^\beta(x, \nabla\phi^\alpha). \tag{2.24}$$

Since $\lambda^\alpha \neq \lambda^\beta$ for $x^1 \neq 0$, it follows from (2.6) that

$$L^\alpha(x, \nabla\phi^\alpha)\, R^\beta(x, \nabla\phi^\alpha) = 0. \tag{2.25}$$

Hence (2.18)–(2.21) reduce to

$$L^\alpha(x, \nabla\phi^\alpha)\, \mathscr{L}(c_0^\alpha(t, x)) = 0, \tag{2.26}$$

$$L^\beta(x, \nabla\phi^\alpha)\, [\mathscr{L}(c_0^\alpha(t, x)) + \gamma_0^{\alpha\beta}(t, x; t)\, R^\beta(x, \nabla\phi^\alpha)] = 0, \tag{2.27}$$

$$L^\beta(x, \nabla\phi^{\alpha\beta})\, \mathscr{L}(c_0^{\alpha\beta}(t, x; \tau)) = 0, \tag{2.28}$$

$$L^\alpha(x, \nabla\phi^{\alpha\beta})\, [\mathscr{L}(c_0^{\alpha\beta}(t, x; \tau)) + \gamma_0^{\alpha\beta\alpha}(t, x; t, \tau)\, R^\alpha(x, \nabla\phi^{\alpha\beta})] = 0. \tag{2.29}$$

We note that (2.26) is identical with (2.7), while (2.27) is an initial condition for $\gamma_0^{\alpha\beta}(\tau, x; \tau)$. Similarly, (2.28) is the usual transport equation for $\gamma_0^{\alpha\beta}(t, x; \tau)$, and (2.29) is an initial condition for $\gamma_0^{\alpha\beta\alpha}(\sigma, x; \sigma, \tau)$. Higher terms in the expansion can be obtained in a similar fashion.

3. Critical Points of the Integrals

The asymptotic behavior of $v(t, x)$ is determined by the critical points of the integrals which appear in (2.11). Here we shall give some properties of these critical points in preparation for the estimate of Section 4 and the interpretation in Section 5.

The single integrals in (2.13) have critical points at the endpoints $\tau = 0$ and $\tau = t$, and also at any intermediate points $\tau_\beta{}^*(t, x)$, given by

$$\phi_\tau^{\alpha\beta}(t, x; \tau_\beta{}^*(t, x)) = 0. \tag{3.1}$$

If $\tau_\beta{}^*$ is not at an endpoint, then an integration by parts shows that the contribution from the endpoint is $0(1/k)$. Moreover, it follows from (2.12), (2.13), and (2.5) that

$$\phi^{\alpha\beta}(t, x; 0) = \phi^\beta(t, x), \tag{3.2}$$

$$\phi^{\alpha\beta}(t, x; t) = \phi^\alpha(t, x). \tag{3.3}$$

Therefore these endpoint contributions have the same form as the first part of (2.11) if $\tau_\beta{}^*$ is not at an endpoint.

Turning now to the stationary points, we first show that $\phi_\tau^{\alpha\beta}$ and $\tau_\beta{}^*$ are constant along rays. After differentiation with respect to τ, (2.12) becomes

$$\phi_{\tau t}^{\alpha\beta} - \sum_{j=1}^{n} \frac{\partial \lambda^\beta}{\partial p_j}(x, \nabla\phi^{\alpha\beta})\, \phi_{\tau x^j}^{\alpha\beta} = 0. \tag{3.4}$$

In view of (2.8), this is the same as $d\phi_\tau^{\alpha\beta}/ds = 0$. Therefore, $\phi_\tau^{\alpha\beta}$ can be evaluated by tracing the ray through (t, x) back to the surface $t = \tau$, and $\tau_\beta{}^*(t, x)$ is constant along rays. By differentiation of (2.13), we obtain

$$\phi_t^{\alpha\beta}(\tau, x; \tau) + \phi_\tau^{\alpha\beta}(\tau, x; \tau) = \phi_t{}^\alpha(\tau, x). \tag{3.5}$$

Substitution from (2.5), (2.12), and (2.23) into (3.4) results in

$$\phi_\tau^{\alpha\beta}(\tau, x; \tau) = \lambda^\alpha(x, \nabla\phi^\alpha(\tau, x)) - \lambda^\beta(x, \nabla\phi^\alpha(\tau, x)). \tag{3.6}$$

According to our assumption (1.3), the right side of (3.6) vanishes at the surface S where $x^1 = 0$, and thus

$$\phi_\tau^{\alpha\beta}(\tau, x; \tau) = 0 \qquad \text{if } x \text{ is on } S. \tag{3.7}$$

We conclude that $\phi_\tau^{\alpha\beta}(t, x; \tau) = 0$ if the ray through (t, x) leaves S at time $t = \tau$, and thus $\tau_\beta{}^*(t, x)$ is the time (if any) at which the ray through (t, x) has left S. If $\tau_\beta{}^*(t, x) = 0$, (3.2) shows that the ray for ϕ^β originates on S. By assumption, the initial data vanish near S, and hence there is no contribution from that endpoint. If $\tau_\beta{}^*(t, x) = t$, then x is on S, and the expansion (2.11) does not simplify.

For later reference, we shall show that $\phi_{\tau\tau}^{\alpha\beta}(\tau, x; \tau) \neq 0$ on S. After differentiation, (3.7) becomes

$$\phi_{t\tau}^{\alpha\beta}(\tau, x; \tau) + \phi_{\tau\tau}^{\alpha\beta}(\tau, x; \tau) = 0 \qquad \text{for } x \text{ on } S. \tag{3.8}$$

Similarly, we obtain

$$\phi_{x^l\tau}^{\alpha\beta}(\tau, x; \tau) = 0 \qquad \text{for } x \text{ on } S, \quad l \geqslant 2. \tag{3.9}$$

After differentiation, (1.3) also yields

$$\lambda_{p_j}^1(x, p) = \lambda_{p_j}^2(x, p) \qquad \text{for } x \text{ on } S, \quad j = 1,..., n. \tag{3.10}$$

When (3.6) is differentiated with respect to x^1 and (3.10) is applied, the result is

$$\phi_{\tau x^1}^{\alpha\beta}(\tau, x; \tau) = \lambda_{x^1}^\alpha(x, \nabla\phi^\alpha) - \lambda_{x^1}^\beta(x, \nabla\phi^\alpha). \tag{3.11}$$

Now we differentiate (2.12) with respect to τ to obtain

$$\phi_{t\tau}^{\alpha\beta}(t, x; \tau) = \sum_{j=1}^{n} \lambda_{p_j}^\beta(x, \nabla\phi^{\alpha\beta}) \frac{\partial^2\phi^{\alpha\beta}(t, x; \tau)}{\partial x^j \partial\tau}. \tag{3.12}$$

If $t = \tau$ and x is on S, we may apply (3.9), (3.11), and (2.23). Therefore

$$\phi_{t\tau}^{\alpha\beta}(\tau, x; \tau) = \lambda_{p_1}^\beta(x, \nabla\phi^\alpha)\,(\lambda_{x^1}^\alpha(x, \nabla\phi^\alpha) - \lambda_{x^1}^\beta(x, \nabla\phi^\alpha)). \tag{3.13}$$

By our hypothesis (1.4), the right side of (3.13) does not vanish. We conclude from (3.7) that $\phi_{\tau\tau}^{\alpha\beta}(\tau, x; \tau)$ does not vanish for x on S. A geometrical argument (which we omit) shows that $\phi_{\tau\tau}^{\alpha\beta}$ does not vanish in any region containing S for which the associated rays have no caustics.

Now we consider the critical points of the double integrals in (2.11).

These might be either interior or boundary points of the domain of integration. An interior stationary point is a point where both $\phi_\sigma^{\alpha\beta\alpha} = 0$ and $\phi_\tau^{\alpha\beta\alpha} = 0$. In analogy with the preceding discussion, it follows from differentiation of (2.14) that $\phi_\sigma^{\alpha\beta\alpha}$ and $\phi_\tau^{\alpha\beta\alpha}$ are constant along rays. The ray for $\phi^{\alpha\beta\alpha}$ consists of three parts (see Fig. 1). The ray begins at $t = 0$, $x = x_0$, and it is determined by (2.8) and (2.9) with $j = \alpha$ and $p = \nabla\phi^\alpha$. The second part begins at $t = \tau$, $x = x_1$. It is determined by (2.8) and (2.9) with $j = \beta$ and $p = \nabla\phi^{\alpha\beta}$. The third part begins at $t = \sigma$, $x = x_2$ with $j = \alpha$ and $p = \nabla\phi^{\alpha\beta\alpha}$.

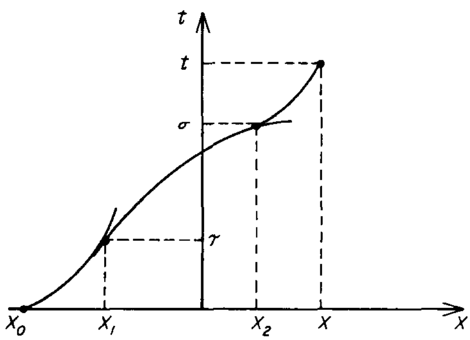

FIGURE 1

The analogy with the preceding discussion of $\phi^{\alpha\beta}$ shows that $\phi_\sigma^{\alpha\beta\alpha} = 0$ if x_2 is on S, i.e.,

$$\phi_\sigma^{\alpha\beta\alpha}(t, x; \tau_\alpha{}^*(t, x), \tau) = 0. \tag{3.14}$$

Similarly, $\phi_\tau^{\alpha\beta\alpha}(t, x; \sigma, \tau) = 0$ if x_1 lies on S. Assuming that rays do not strike S more than once, we conclude that an interior stationary point is possible only if $x_1 = x_2$ and $\sigma = \tau$. But the line $\sigma = \tau$ is part of the boundary of the region of integration in (2.11).

The simplest boundary critical points are points where the tangential derivative of ϕ vanishes. On the line $\tau = 0$, such a point appears at $\sigma = \tau_\alpha{}^*(t, x)$. It can be shown that such points make a contribution of order $k^{-3/2}$ (see Bleistein and Handelsman [1, p. 447]). Similarly, we may neglect the contribution from the point $\sigma = t$, $\tau = \tau_\beta{}^*(t, x)$. The line $\sigma = \tau$ consists entirely of critical points, since from (2.12)–(2.15),

$$\phi^{\alpha\beta\alpha}(t, x; \sigma, \sigma) = \phi^\alpha(t, x). \tag{3.15}$$

The contribution from this segment is studied in the following section.

The corners are also critical points, but their contribution is $O(k^{-2})$ unless they are also stationary points. It is easily seen that a corner is a stationary point only if the ray starts or ends on S. We have assumed zero initial amplitude for rays which start on S, so that this possibility does not occur in the present case. On the other hand, if x is near S, such contributions are important, and the present analysis does not apply.

For later reference, we show that the Hessian matrix of $\phi^{\alpha\beta\alpha}$ does not vanish at the stationary point. It follows from differentiation of (3.15) that

$$\phi_{\sigma\sigma}^{\alpha\beta\alpha} + 2\phi_{\sigma\tau}^{\alpha\beta\alpha} + \phi_{\tau\tau}^{\alpha\beta\alpha} = 0 \qquad \text{if} \qquad \sigma = \tau. \tag{3.16}$$

Hence, the Hessian Δ is given by

$$\Delta = \phi_{\sigma\sigma}^{\alpha\beta\alpha}\phi_{\tau\tau}^{\alpha\beta\alpha} - (\phi_{\sigma\tau}^{\alpha\beta\alpha})^2 = -(\phi_{\sigma\sigma}^{\alpha\beta\alpha} + \phi_{\sigma\tau}^{\alpha\beta\alpha})^2. \tag{3.17}$$

The preceding discussion (see (3.14)) shows that at the stationary point, we must have

$$\phi_{\sigma}^{\alpha\beta\alpha}(t, x, \tau_\alpha{}^*(t, x), \tau_\alpha{}^*(t, x)) = 0. \tag{3.18}$$

After differentiation with respect to t, we have

$$\phi_{\sigma t}^{\alpha\beta\alpha} + (\phi_{\sigma\tau}^{\alpha\beta\alpha} + \phi_{\sigma\sigma}^{\alpha\beta\alpha}) \frac{\partial \tau_\alpha{}^*}{\partial t} = 0 \tag{3.19}$$

at the stationary point. By our previous argument (see (3.13)), $\phi_{t\sigma}^{\alpha\beta\alpha} \neq 0$ near S, and hence $\Delta \neq 0$ as well.

4. Estimation of the Double Integral

In this section, the asymptotic behavior of the integral

$$I = \iint_D c(\sigma, \tau)\, e^{ik\phi(\sigma,\tau)}\, d\sigma\, d\tau, \tag{4.1}$$

is studied for large k. Here D is a bounded domain whose boundary consists of three curves γ_i with the representations $f_i(\sigma, \tau) = 0$ ($i = 1, 2, 3$). It is assumed that ϕ is constant along the curve $f_1(\sigma, \tau) = 0$, $\nabla\phi$ vanishes at only one point (σ_0, τ_0) on this curve, and there are no other points in D at which $\nabla\phi$ vanishes. It is further assumed that the Hessian Δ of ϕ is different from zero at (σ_0, τ_0). Without loss of generality, we can choose $\phi = 0$ on γ_1. The main result of this section is that

$$I = \frac{1}{ik}\left\{P\int_{\gamma_1} \frac{c(\sigma, \tau)}{\partial\phi/\partial n}\, ds - i\pi \frac{c(\sigma_0, \tau_0)}{\Delta(\sigma_0, \tau_0)}\right\} + 0(k^{-3/2}), \tag{4.2}$$

if σ_0, τ_0 is not near a corner of D, and that $I = 0(\log k/k)$ uniformly. In (4.2), $P \int$ denotes the Cauchy principal value of the integral. If $\partial\phi/\partial n$ did not vanish on γ_1, then the integral in (4.2) would be the usual result. The rule, in case ϕ_n does vanish, is to take the principal value of the integral and subtract one-half of the contribution from the pole. This is similar to the integral of the generalized function $(x + i0)^{-1}$ (see Gelfand and Shilov [3, pp. 34, 60]).

Let D^* be a neighborhood of σ_0, τ_0. Further conditions on D^* are given below. Then the integral I can be expressed as $I = I_1 + I_2$, where

$$I_1 = \iint_{D-D^*} ce^{ik\phi} \, d\sigma \, d\tau, \tag{4.3}$$

$$I_2 = \iint_{D^*} ce^{ik\phi} \, d\sigma \, d\tau. \tag{4.4}$$

Since $\nabla\phi$ does not vanish in $D - D^*$, we choose a constant vector A such that $A \cdot \nabla\phi$ does not vanish in $D - D^*$ (for example, let one component of A be real and the other be complex). Then I_1 can be expressed as

$$I_1 = \frac{1}{ik} \iint_{D-D^*} \frac{c}{A \cdot \nabla\phi} \nabla \cdot (Ae^{ik\phi}) \, d\sigma \, d\tau. \tag{4.5}$$

An application of Gauss' theorem yields

$$I_1 = \frac{1}{ik} \int_{\partial(D-D^*)} \frac{cn \cdot A}{A \cdot \nabla\phi} \, ds - \frac{1}{ik} \iint_{D-D^*} \nabla \left(\frac{c}{A \cdot \nabla\phi} \right) \cdot Ae^{ik\phi} \, d\sigma \, d\tau. \tag{4.6}$$

It is clear from (4.6) that $I_1 = 0(1/k)$. Since the double integral in (4.6) has the same form as I_1, it must also have order $1/k$. Therefore, the second term in (4.6) has order $1/k^2$. The portion of the line integral along γ_2 and γ_3 can be shown to have order $k^{-1/2}$ by the method of stationary phase. Therefore

$$I_1 = \frac{1}{ik} \int_{\tilde{\gamma}_1} \frac{cn \cdot A}{A \cdot \nabla\phi} \, ds + 0(k^{-3/2}), \tag{4.7}$$

where $\tilde{\gamma}_1$ denotes the portion of γ_1 contained in $\partial(D - D^*)$. Since ϕ is constant along $\tilde{\gamma}_1$, $A \cdot \nabla\phi = (n \cdot A)\phi_n$, and the integrand of (4.7) can be written as c/ϕ_n.

Now we turn to estimation of I_2. First we assume that the critical point (σ_0, τ_0) is not near a corner of D. New regular coordinates r and s are introduced such that $\phi = rs$. The Jacobian of the transformation at (σ_0, τ_0) is the Hessian of ϕ at that point. Further details are given in Bleistein and

Handelsman [1, pp. 438–440]. Let r be the coordinate which vanishes on γ_1. We choose D^* to be the domain which is mapped into the rectangle $0 \leqslant r \leqslant b$, $-a \leqslant s \leqslant a$, where a and b are so small that the Jacobian is bounded away from zero in D^*. With this change of variables, (4.4) becomes

$$I_2 = \int_0^b \int_{-a}^a e^{ikrs} g(r, s)\, ds\, dr, \tag{4.8}$$

where

$$g(r, s) = c(\sigma, \tau)\, [J(\sigma, \tau)]^{-1}.$$

We now write $I_2 = I_3 + I_4 + I_5$, where

$$I_3 = \int_0^\beta \int_{-a}^a e^{ikrs}(g(r, s) - g(0, s))\, ds\, dr, \tag{4.9}$$

$$I_4 = \int_0^b \int_{-a}^a e^{ikrs}(g(0, s) - g(0, 0))\, ds\, dr, \tag{4.10}$$

$$I_5 = \int_0^b \int_{-a}^a e^{ikrs} g(0, 0)\, ds\, dr. \tag{4.11}$$

After performing the r integration, I_3 becomes

$$I_4 = \frac{1}{ik} \int_{-a}^a e^{ikbs} \frac{g(0, s) - g(0, 0)}{s}\, ds - \frac{1}{ik} \int_{-a}^a \frac{g(0, s) - g(0, 0)}{s}\, ds. \tag{4.12}$$

The first integral in (4.12) has order $1/k$ (after integration by parts). Because of the symmetry in the interval of integration, the second integral is a principal value integral. Thus (4.12) becomes

$$I_4 = -\frac{1}{ik} P \int_{-a}^a \frac{g(0, s)}{s}\, ds + 0\left(\frac{1}{k^2}\right). \tag{4.13}$$

In (4.11), integration with respect to s and manipulation of the integration limits yields

$$I_5 = \frac{2}{k} g(0, 0) \left[\frac{\pi}{2} - \int_b^\infty \frac{\sin kar}{r}\, dr\right]. \tag{4.14}$$

The integral in (4.14) has order $1/k$, and thus

$$I_5 = \frac{\pi g(0, 0)}{k} + 0\left(\frac{1}{k^2}\right). \tag{4.15}$$

Integration by parts in (4.9) yields

$$\begin{aligned} I_3 = {} & \frac{1}{ik} \int_0^b e^{ikrs} \frac{g(r, s) - g(0, s)}{r} \Big|_{s=-a}^{s=a} dr \\ & - \frac{1}{ik} \int_0^b \int_{-s}^{s} e^{ikrs} \frac{g_s(r, s) - g_s(0, s)}{r}\, ds\, dr, \end{aligned} \tag{4.16}$$

and thus $I_3 = 0(1/k)$. It follows from (4.15), (4.15), and (4.16) that $I_2 = 0(1/k)$. The double integral in (4.16) has the same form as I_2 and therefore has order $1/k$. The single integrals have order $1/k$, and thus $I_3 = 0(1/k^2)$. This estimate, together with (4.13) and (4.15), implies (4.2).

In case (σ_0, τ_0) is near a corner of D, we make the same change of variables as before. Then I_2 has the form

$$I_2 = \int_0^b \int_a^{h(r)} e^{ikrs} g(r, s)\, ds\, dr. \tag{4.17}$$

An integration by parts yields

$$\begin{aligned} ikI_2 = \int_0^b \frac{e^{ikrh(r)} - 1}{r} g(r, h)\, dr - \int_0^b \frac{e^{ikra} - 1}{r} g(r, a)\, dr \\ + \int_0^b \int_a^{h(r)} \frac{e^{ikrs} - 1}{r} g_s(r, s)\, ds\, dr. \end{aligned} \tag{4.18}$$

This immediately yields the estimate

$$\begin{aligned} |\, kI_2 \,| \leqslant \max_{D^*} |\, g \,| \int_0^b \left\{ \left| \frac{e^{ikrh(r)} - 1}{r} \right| + \left| \frac{e^{ikra} - 1}{r} \right| \right\} dr \\ + \max_{D^*} |\, g_s \,| \int_0^b \int_0^{\hat{h}} \left| \frac{e^{ikrs} - 1}{r} \right| ds\, dr, \end{aligned} \tag{4.19}$$

where

$$\hat{h} = \max_{0 \leqslant r \leqslant b} h(r).$$

If the interval of integration is broken into $[0, b/k] + [b/k, b]$, then it follows that

$$\int_0^b \left| \frac{e^{ikrh(r)} - 1}{r} \right| dr \leqslant 2b \max_{0 \leqslant r \leqslant b} |\, h(r) | + 2 \log k. \tag{4.20}$$

All of the integrals with respect to r in (4.19) have the same form as (4.20). Thus we obtain

$$|\, I_2 \,| \leqslant k^{-1} \,|\, a_1 + a_2 \log k \,|, \tag{4.21}$$

where a_1 and a_2 are independent of k. Hence I has order $k^{-1} \log k$ when (σ_0, τ_0) is near a corner. This result is also valid when the critical point (which lies on the continuation of γ_1) is not in D. If (σ_0, τ_0) is not in D and not near a corner of D, then the previous argument shows that

$$I = \frac{1}{ik} \int_{\gamma_1} \frac{c(\sigma, \tau)}{\phi_n(\sigma, \tau)} ds + 0(k^{-3/2}). \tag{4.22}$$

These results can be applied to the approximate solution (2.11) as follows: we denote the exact solution of (1.1) and (1.5) truncated after 1 term by $v_0(t, x)$. The construction of Section 2 shows that

$$\mathscr{L}(v - v_0) = \int_0^t \int_0^\sigma \sum_{\substack{\alpha=1 \\ \beta=3-\alpha}}^{2} \exp(ik\phi^{\alpha\beta\alpha})\, c_0^{\alpha\beta\alpha}\, d\tau\, d\sigma + 0\left(\frac{1}{k}\right). \tag{4.23}$$

Now applying our estimate to (4.23), we obtain

$$\mathscr{L}(v - v_0) = 0\left(\frac{\log k}{k}\right). \tag{4.24}$$

For a symmetric system, the energy inequality (see Courant [2, Chap. VI]) yields

$$\begin{aligned} v(t, x) = {} & \sum_{j=1}^{m} c_0{}^j(t, x) \exp(ik\phi^j(t, x)) \\ & + \sum_{\substack{\alpha=1 \\ \beta=3-\alpha}}^{2} \int_0^t c_0^{\alpha\beta}(t, x; \tau) \exp(ik\phi^{\alpha\beta}(t, x; \tau))\, d\tau + 0\left(\frac{\log k}{k}\right), \end{aligned} \tag{4.25}$$

where the order symbol must be taken in the L_2 sense.

5. Interpretation of the Results

In this section we shall simplify and interpret the integrals in (4.25). The discussion of Section 3 shows that if the integral has a stationary point, it occurs at $\tau_\beta{}^*(t, x)$, the point where the ray through (t, x) meets S. Moreover, $\phi_{\tau\tau}^{\alpha\beta}(t, x; \tau_\beta{}^*) \neq 0$ if (t, x) is not too far from S. Hence, the method of stationary phase is applicable, and we obtain

$$\begin{aligned} v(t, x) = {} & \sum_{j=1}^{m} c_0{}^j(t, x) \exp(ik\phi^j(t, x)) \\ & + \sum_{\substack{\alpha=1 \\ \beta=3-\alpha}}^{2} R_{\alpha\beta} \frac{c_0^{\alpha\beta}(t, x; \tau_\beta{}^*)}{|\phi_{\tau\tau}^{\alpha\beta}(t, x, \tau_\beta{}^*)|^{1/2}} \exp(ik\phi^{\alpha\beta}(t, x; \tau_\beta{}^*)). \end{aligned} \tag{5.1}$$

Here $R_{\alpha\beta}$ plays the role of a diffraction coefficient; it is given by

$$R_{\alpha\beta} = \left\{ 2\pi k^{-1} \exp \left[i \frac{\pi}{4} \operatorname{sgn}(\phi_{\tau\tau}^{\alpha\beta}(t, x; \tau_\beta^*)) \right] \right\}^{1/2}. \tag{5.2}$$

The phase $\phi^{\alpha\beta}(t, x; \tau_\beta^*)$ corresponds to a ray (2.8), (2.9) with $j = \alpha$ which, upon meeting S, is converted into a ray with $j = \beta$. Indeed, it follows from (2.12) and (3.1) that

$$\frac{\partial}{\partial t} \phi^{\alpha\beta}(t, x; \tau_\beta^*(t, x)) = \lambda^\beta(t, x; \nabla\phi^{\alpha\beta}(t, x; \tau_\beta^*(t, x))). \tag{5.3}$$

In order to complete the geometrical interpretation of (4.25), we should verify that the coefficient $c_0^{\alpha\beta} \mid \phi_{\tau\tau}^{\alpha\beta} \mid^{-1/2}$ satisfies the transport equation which corresponds to (5.3). This is not the same as (2.28), since τ does not depend upon (t, x) in (2.28). We omit this verification, since it is given in slightly different notation in Ludwig [6, pp. 246–247]. The initial condition for $c_0^{\alpha\beta} \mid \phi_{\tau\tau}^{\alpha\beta} \mid^{-1/2}$ is given by (2.27).

Finally, in Fig. 2, we sketch the regions in which the various terms are present. In order to distinguish them, rays for $j = 1$ are drawn concave downwards, while rays for $j = 2$ are drawn concave upwards. Regions I, II, and III are distinguished by drawing rays emanating from S at $t = 0$. In region I, neither ray through P_{I} meets S, and therefore a solution of the form (2.1) is valid. In region II, a point P_{II} is reached by direct rays originating at A and B, and also by a ray ($j = 2$) which gives rise to a ray ($j = 1$) at S. In region III, points are reached by four different rays, in obvious analogy. As was remarked previously, the expression (5.1) is not valid at the boundaries between I and II and between II and III unless the initial data vanish near S.

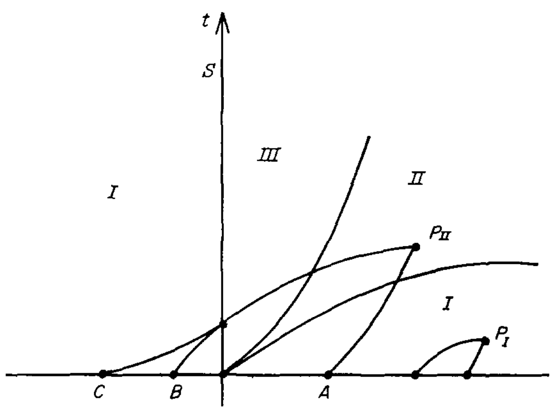

FIGURE 2

6. An Application to MHD

We consider the Lundquist equations (see Jeffrey and Taniuti [5]), assuming isentropic flow, and linearizing about a basic flow with constant magnetic field in the direction i_1. We assume further that the density ρ_0 and the gravitational field depend only upon x^1. Such conditions may be present in the sun, and the present analysis might explain some chromospheric heating in the presence of a magnetic field (See Howe [4].) The linearized equations are then

$$\begin{gathered} H_t - \nabla \times (u \times i_1) H_0 = 0, \\ \rho_t + \rho_0 \nabla \cdot u + u^1 \rho_{0x^1} = 0, \\ \rho_0 u_t + \nabla a^2 \rho - \mu(\nabla \times H) \times i_1 H_0 = \rho g i_1, \qquad \nabla \cdot H = 0, \end{gathered} \tag{6.1}$$

where

$$a = (P_\rho(\rho)\,|_{\rho=\rho_0})^{1/2}$$

is the sound speed. We can put this system in symmetric form by introducing the change of variables

$$\begin{aligned} H &= \sqrt{\rho_0/\mu}\, M, \\ H_0 &= \sqrt{\rho_0/\mu}\, M_0, \\ \rho &= (\rho_0/\mu)\, d. \end{aligned} \tag{6.2}$$

This yields when written in matrix form

$$LV \equiv V_t + \sum A^\nu V_{x^\nu} + BV = 0,$$

where

$$V = \begin{Bmatrix} M \\ u \\ d \end{Bmatrix},$$

$$A^1_{25} = A^1_{36} = -A^2_{15} = -A^3_{16} = \frac{2}{e} B_{52} = \frac{2}{e} B_{63} = -M_0, \qquad e = \rho_{0x^1}/\rho_0,$$

$$A^1_{47} = A^2_{57} = A^3_{67} = \frac{1}{e} B_{74} = a,$$

$$B_{47} = a_{x^1},$$

$$A^k_{ji} = A^k_{ji}, \qquad k = 1, 2, 3,$$

$$A^k_{ji} = B_{ij} = 0 \qquad \text{elsewhere.}$$

For propagation in the x^1 direction the characteristic determinant is

$$\det | I\phi_t + A^1\phi_{x^1} | = 0.$$

This equation yields seven characteristic functions ϕ^i which satisfy

$$\phi_t^{\,i} + (-1)^{i+1} a(x^1)\, \phi^i_{x^1} = 0, \qquad i = 1, 4,$$

$$\begin{pmatrix} \phi^i \\ \phi^{i+1} \end{pmatrix}_t + (-1)^{i+1} M_0(x^1) \begin{pmatrix} \phi^i \\ \phi^{i+1} \end{pmatrix}_{x^1} = 0, \qquad i = 2, 5,$$

$$\phi_t^{\,7} = 0.$$

The components of the right null vector r^i of a $(I\phi_t^{\,i} + A^1\phi^1_{x^1})$ are

$$r_4^{\,i} = (-1)^{i+1} r_7^{\,i} = r_2^{i+1} = (-1)^i r_5^{i+1} = r_3^{i+2} = (-1)^i r_6^{i+2} = r_1^{\,7} = (\tfrac{1}{2})^{1/2},$$

$$i = 1, 4, \qquad r_i^{\,j} = 0$$

elsewhere.

By symmetry the components of the left null vectors l^i are $l_j^{\,i} = r_j^{\,i}$. We now suppose that there exists a point say $x^1 = 0$ where the Alfvèn speed equals the sound speed. Moreover, we assume that $M_{0x^1}(0) \neq a_{x^1}(0)$, $M(0) = a(0) \neq 0$ and that $\phi_x^{\,i}(0, t) \neq 0$, for $i = 1, 2$. Then the propagation speeds cross at $x^1 = 0$. We remark that if the density of the fluid decreases sufficiently rapidly with increasing altitude (x^1), the sound speed a decreases, while the Alfvèn speed M_0 increases with increasing altitude. If at some altitude the sound speed is greater than the Alfvèn speed, then at some higher altitude these two speeds will be equal.

The results of the previous section are not immediately applicable to system (5.2) because of the multiple characteristics. However, Ansatz (2.11) can be inserted into (5.2) and the resulting equation can be solved by a procedure similar to that in Section II. The major difference is that there exist two independent transport equations associated with propagation in the Alfvèn modes. We give results for propagation associated with coupling of modes 1, 2 and 3. The first order results are

$$V \sim \delta^1 r^1 e^{ik\phi^1} + R^{12}(\delta^{12} r^2 + \delta^{13} r^3)\, e^{ik\phi^{12}} + (\delta^2 r^2 + \delta^3 r^3)\, e^{ik\phi^2}$$

$$+ R^{21}\delta^{21} r^1 e^{ik\phi^{21}} + 0\left(\frac{\log k}{k}\right), \qquad x, t \in D^1 \cap D^2,$$

where D^i denotes the cylindrical region bounded by the surface $x^1 = 0$ and the rays of ϕ^i emanating from $x^1 = t = 0$, and

$$\phi_t^{ij} = U^j(x^1)\,\phi_{x^1}^{ij},$$

$$\phi^{ij} = \phi^i, \qquad x_1 = 0,$$

$$\delta_t^{\,i} + U^i\delta_{x^1}^i + s^i\delta = 0, \qquad \delta_t^{ij} + U^i\delta_{x^1}^{ij} + s^j\delta = 0,$$

$$\delta^{ij} = \tfrac{1}{2}\,|\,d^{ij}\,|^{-1/2}\,a\delta_{x_1{}^j}^{\,i}, \qquad x^1 = 0,$$

$$R^{ij} = \left(\frac{2\pi}{k}\right)^{1/2} \exp\left(\frac{\pi i}{4}\operatorname{sgn} d^{ij}\right),$$

$$d^{ij} = U^i(0)\,(U_{x^1}^j(0) - U_{x^1}^i(0))\,\phi_{x^1}^i(0, t),$$

$$s^1 = \tfrac{1}{2}\,(ea + a_{x^1}), \qquad s^2 = s^3 = +\,\tfrac{1}{4}\,eU^2,$$

$$U^1 = a, \qquad U^2 = U^3 = M_0\,.$$

We remark that coupling associated with modes 4, 5, and 6 can be found formally from the above result by replacing U^i, s^1 by $-\,U^i$, $-\,s^1$ respectively whenever they appear and then adding three to all the superscripts.

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